Garner Cochran's Reseach Statement

My research area is graph theory and combinatorics. In particular I am interested in diameter problems on graphs. I received my undergraduate degree from Trinity University in San Antonio, Texas in 2013. I was introduced to research early on in my undergraduate through completing a project titled "Generating functions and Wilf equivalence for generalized interval embeddings" with Professor Brian Miceli, resulting in a publication. After my undergraduate studies, I entered the PhD program at the University of South Carolina (USC) under the direction of Éva Czabarka studying diameter problems on graphs.

I have had the opportunity to carry out collaborative work in various workshops and through an international reseach collaboration. I have gained experience in applying for funding through writing various proposals, a number of which got support. In the Spring of 2017, I submitted a grant proposal for the SPARC Fellowship to the Office of the Vice President for Research at USC. I was awarded this grant to travel and continue research in diameter problems on graphs with Professor Peter Dankelmann, one of the experts in the field, at the University of Johannesburg in South Africa. I also applied for and have been a fully funded participant at a number of workshops which have led to ongoing collaborations. I participated twice in the Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics in Summer 2015 and Summer 2016. During these workshops, I participated on projects involving enumerating RNA foldings on trees and proving relationships for d-matching polynomials using combinatorial proof techniques. The first project is leading to a paper that will be submitted imminently, while we are currently preparing a paper from the second for submission.

I also participated in the Mathematical Research Communities workshop titled "Beyond Planarity: Crossing Numbers of Graphs" in Summer 2017. I participated on a project involving trying to resolve the Zarankiewicz conjecture with additional hypotheses. I was selected by the workshop organizers to be the lead organizer for the special session related to this workshop that is being held at the Joint Mathematics Meetings in January 2018. This position has involved inviting and corresponding with speakers, as well as scheduling the talks for the session. I was also tasked with organizing the USC Discrete Math group's weekly seminar during the 2014-2015 academic year. This involved similar responsibilities to the MRC session organization. I was a fully funded participant in the Summer School for Network Science at USC in Summer 2013.

Through my collaborations, I have had the opportunity to work with mathematicians from many fields, computer scientists, and biologists. This has given me the opportunity to learn how to disseminate mathematical information to groups of different backgrounds. My projects have mainly fallen into the categories of diameter problems on graphs, applications of combinatorics to biology, and proving generating function identities using combinatorial proofs.

1. Diameter Problems on Graphs

Diameter problems on graphs have been well studied by graph theorists. They are of particular importance to applied problems involving social and transportation networks, as distance is an important metric to measure speed in which objects can travel in real world networks.

1.1. **The Oriented Diameter of a Bridgeless Graph Given Certain Parameters.** In unoriented graphs one can traverse the connections between the vertices in any direction, a model that is clearly not appropriate for all applications. Oriented graphs have the direction of traverse specified on these connections; and this changes the nature of the problem of finding the (oriented) distance and diameter in a network. The question of whether a bridgeless unoriented graph of small diameter has an orientation with small oriented diameter has been thoroughly investigated [5, 6, 15]. Many results have also been found about the oriented diameter of certain classes of graphs [10, 12, 13, 18].

A result by Erdős, Pach, Pollack, and Tuza [8] gives us that the diameter of a connected unoriented graph of order n and minimum degree δ is $\frac{3}{\delta+1}n + O(1)$. In my current research, I am working on finding upper bounds of a similar form on the best possible oriented diameter of a strong orientation of a bridgeless graph. A 2015 paper of Prof. Dankelmann and Dr. Sheng Bau in the European Journal of Combinatorics [2] proved that the best diameter of such orientation is at most $11\frac{n}{\delta+1}+9$. We focused on strengthening lower and upper bounds on the the best diameter of such an orientation. We improved the bound cited above to $5\frac{n}{\delta-1}+120$ and are currently preparing this result for submission. I also see some slack in this proof that may allow us to push the upper bound closer to the lower bound.

Another important parameter is the domination number, γ, of a graph. It is the minimum number of vertices in the graph so that every vertex in the graph is either one of these vertices or is directly connected to one of these vertices. For more information on the domination number, consider this survey by Goddard and Henning [9]. This parameter and its variants are much studied, as they have important applications in social networks [1,4]. We also consider the girth g of a graph, which is the size of its smallest cycle. We are working on refining the bounds for oriented diameter above by involving the girth and the domination number. I have made significant progress on the problem involving the girth.

1.2. **Size condition for diameter 2 orientability of a graph.** Given the complete graph, K_n, it is well known that for any $n \geq 5$ there exists an orientation of the edges of the graph such that the diameter of this orientation is 2. In a survey by Koh and Tay [14], Problem 4 asks how many edges can we remove from K_n such that no matter the set of edges removed, there exists an orientation of G, $\overrightarrow{G}$, such that $\operatorname{diam}(\overrightarrow{G}) \leq 2$. It is conjectured that for $n \geq 5$, you may remove any collection of $n-5$ edges and find a diameter 2 orientation of the graph. Éva Czabarka, László Székely, Peter Dankelmann and myself are in the process of writing up a result showing this conjecture is true. Our results use a combination of constructions reducing the cases we need to check to a finite subset of problems for which a computer search is sufficient.

Further directions on this problem that I plan to work on include adding a further condition that the minimum degree of the graph must be at least k as suggested in the survey [14]. We are also working on extending this problem to consider the maximum number of edges that can be removed such that $\operatorname{diam}(\overrightarrow{G}) \leq 3$, and then generalizing this result to any k. That is, how large can a function $f(k,n)$ be such that removing less than $f(k,n)$ edges from K_n, we can find an orientation of G, $\overrightarrow{G}$, such that $\operatorname{diam}(\overrightarrow{G}) \leq k$.

2. Applications of Combinatorics to Biology: Combinatorial Aspects of RNA Folding

The molecule ribonucleic acid (RNA) consists of a single strand of the four nucleotides adenine, uracil, cytosine, and guanine. In short, RNA is representable by finite sequences (or words) from the alphabet A, U, C, and G, lending itself to combinatorial study. In contrast to the double helix of DNA, the single-stranded nature of RNA often results in RNA folding onto itself as the nucleotides form bonds. As in DNA, we have the Watson-Crick pairs so that C and G form bonds and A and U form bonds. However, RNA has one more bond that may form, the wobble pair G and U. Predicting the folded structure of RNA is of great interest, as the folded structure gives indication of its functionality. In this problem, we directed our attention to a generalized combinatorial model, motivated by the folding of RNA. This model was first introduced by Black, Drellich, and Tymoczko [3] with an initial restriction made to the Watson-Crick bonding pairs, $A-U$ and $G-C$. We generalized bonding pairs to an alphabet of size m, $A_1,\dots,A_m$ and their complimentary pairs $\overline{A}_1,\dots,\overline{A}_m$. The RNA example is an alphabet of size $m=2$, where $A_1 = A$, $A_2 = G$, $\overline{A}_1 = U$, and $\overline{A}_2 = C$.

In this problem, consider rooted plane trees and order the half edges of the rooted plane tree by starting at the right and tracing the perimeter counterclockwise. Given a word w, we call a tree w-valid if for each edge of T, the two letters from w which are labels of that edge are complements. We call a word foldable if there is a plane tree that is w-valid. We call a word w k-foldable if there are exactly k plane trees that are w-valid.

In this problem we asked the questions of enumerating the words of a given length that are 1-foldable. We also examined the set $\mathcal{R}(n,m)$ consisting of all integers k for which there exists a word of length $2n$ with exactly k w-valid trees.

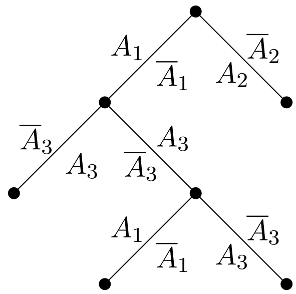


FIGURE 2.1. A folding of $d(w) = A_1\overline{A}_3A_3\overline{A}_3A_1\overline{A}_1A_3\overline{A}_3A_3\overline{A}_1A_2\overline{A}_2$ on a tree.

There are many extensions available for research on this problem that are suitable for undergraduate research as the objects are accessible enough that with some work, examples, counterexamples, and simple proofs can be found to improve bounds. For further work, we are interested in considering similar problems allowing the wobble pair $G-U$ in addition to the normal base pairs, $A-U$ and $G-C$, and the smallest value k for which $k \notin \mathcal{R}(n,m)$.

3. Proving Generating Function Identities Using Combinatorial Proof

3.1. **Generating Functions and Wilf Equivalence for Generalized Interval Embeddings.** Different kinds of embeddings of a smaller word into a larger word have been well studied(citations). For us, we consider words that are simply a string of letters from the alphabet $\mathbb{N}$. Let w be a word from this alphabet, and let $|w|$ denote the number of letters in w, $w = w_1 \dots w_{|w|}$, and $\sum w = \sum_{i=1}^{|w|} w_i$. Define the weight of w to be $wt(w) = t^{|w|}x^{\sum w}$. As an example, see that for $w = 1443$, $wt(w) = t^4x^{12}$. Noonan and Zeilberger [17] define u to be a factor of w if there is a string, v, of $|u|$ consecutive letters in w such that for all $1 \le i \le |u|$, $u_i = v_i$. For example, if $u = 1443$ and $w = 841443117$, then w has u as a factor since $w_3w_4w_5w_6 = u$. Later this was generalized by Kitaev, Liese, Remmel, and Sagan [11]. They consider a word w to embed u if there is a string, v, of $|u|$ consecutive letters in w such that for all $1 \le i \le |u|$, $u_i \le v_i$. For example, if $u = 154$ and $w = 16563$, then w embeds u into two places, $\underline{165}63$ and $1\underline{656}3$. These substrings are said to be embeddings of u into w. Let $\mathcal{E}(u)$ be the set of all words w for which u is embedded in w. We define the infinite generating series for the words that avoid u, $\mathcal{E}(u)$, in the variables t and x to be $\sum_{w \in \mathcal{E}(u)} wt(w)$. We define two words u and v to be Wilf equivalent, denoted $u \sim v$, if

$$\sum_{w \in \mathcal{E}(u)} wt(w) = \sum_{w \in \mathcal{E}(v)} wt(w).$$

The rearrangement conjecture states if $u \sim v$, then u and v must be rearrangements of each other. Let $A = (a_1, a_2, \dots, a_{|u|})$. We studied a generalized version of this problem called interval embedding where instead of needing the condition $u_i \le v_i < \infty$ for all v_i for a word to be embeded, we say w embeds u^A if there exists $|u|$ consecutive letters such that for all $1 \le i \le |u|$, $u_i \le w_i \le u_i + a_i$. We proved a similar statement to that of the rearrangement conjecture, as long as one of the $a_i < \infty$.

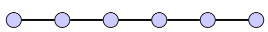
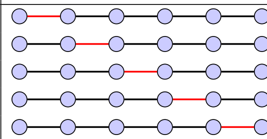
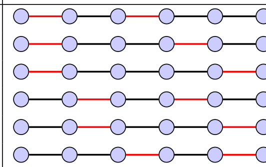
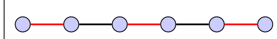
I completed this problem in my undergraduate thesis, and it helped me understand how, with minor adaptations, to make difficult problems accessible to undergraduates.

3.2. ***d*-Matching Polynomials.** In the area of algebraic graph theory, many different kinds of graph polynomials have been studied. For examples, consider the chromatic polynomial (sometimes called the Tutte polynomial) [7], and the matching polynomial [16]. In this project, we studied the d-matching polynomial of a graph G.

Let $M \subseteq E(G)$ be a matching. Let $a(G, m)$ denote the number of matchings M of G with $|M| = m$. For a graph G with $|V(G)| = n$, the **matching polynomial**, $\mathcal{M}_G(x)$ is defined by

$$\mathcal{M}_G(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m a(G, m) x^{n-2m}.$$

Consider the example $G = P_6$, the path graph on 6 vertices.

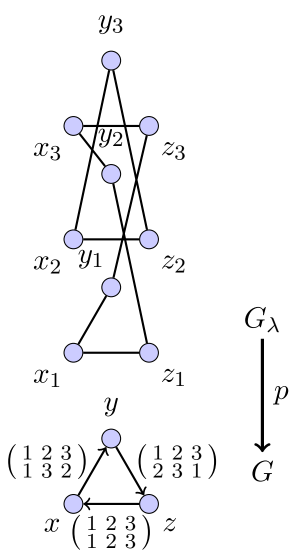
m	0	1	2	3
Matchings of size m				
$a(P_6, m)$	1	5	6	1

Therefore, $\mathcal{M}_{P_6}(x) = x^6 - 5x^4 + 6x^2 - 1$.

If we let C and G be graphs with respective vertex sets $V(C)$ and $V(G)$ and edge sets $E(C)$ and $E(G)$, a surjective map $p : V(C) \to V(G)$ is a **covering map** if for every $v \in V(C)$, edges incident to v correspond bijectively to edges incident to $p(v) \in V(G)$. By definition, we have that $|p^{-1}(v)| = |p^{-1}(w)|$ for all $v, w \in V(G)$. If $|p^{-1}(v)| = d$, then we say C is a d-**covering** of G. Let d be a positive integer and denote by S_d the group of permutations of $[d]$. Assign to each edge of G an orientation. For an edge $e \in E(G)$, call $t(e)$ and $h(e)$ the head and tail respectively. We define $\mathcal{L}_d(G)$ to be the set of all S_d labellings of G. More simply, this is the set of all assignments $\phi : E(G) \to S_d$. Let $\phi(e) := \lambda_e$. Given $\lambda \in \mathcal{L}_d(G)$, we define the unique d-covering G_λ of G as follows. Let $V(G_\lambda) := V(G) \times \{1, \ldots, d\}$. For convenience, we write $(v, i) \in V(G)$ as v_i. The edges in G_λ are formed by connecting $h(e)_{\lambda_e(i)}$ to $t(e)_i$ for every $i \in \{1, \ldots, d\}$. Explicitly,

$$E(G_\lambda) = \{(t(e)_i, h(e)_{\lambda_{e(i)}}) : e \in E(G),\ 1 \leq i \leq d\}.$$

Consider the following example of a 3 covering of the 3-cycle associated with an assignment of permutations to the edges of the cycle.



Definition 3.1. Let G be a finite, undirected graph. The d**-matching polynomial of** G is defined by

$$\mathcal{M}_{G,d}(x) = \frac{1}{|\mathcal{L}_d(G)|} \sum_{\lambda \in \mathcal{L}_d(G)} \mathcal{M}_{G_\lambda}(x).$$

We investigated a conjecture of Hall that $\mathcal{M}_{C_{n,d}}(x) = \mathcal{M}_{P_{nd+n-1}}(x)/\mathcal{M}_{P_{n-1}}(x)$ (citation). It is well known that there is a relationship between the classical matching polynomials and the Chebyshev polynomials, namely: $\mathcal{M}_{P_n}(2x) = \mathcal{U}_n(x)$ and $\mathcal{M}_{C_n}(2x) = 2\mathcal{T}_n(x)$ (citation). We used this to rewrite Hall's conjecture into an equivalent, but more combinatorially friendly form: $\mathcal{M}_{C_{n,d}}(x) = \mathcal{M}_{P_d}(\mathcal{M}_{C_n}(x))$. We then proved this equality using a combination of algebraic and combinatorial proof techniques. We are working on generalizing these results to larger classes of graphs. Our next step is to investigate graphs G which are cycles with one chord.

4. Zarankiewicz problem

This problem was a collaboration begun at the Mathematical Research Communities program on Crossing Numbers. This problem is more well known as Turán's brick factory problem. Zarankeiwicz and Urbanic found what were thought to be independent proofs in 1954 and 1955. Their results were published for 11 years until Kainen and Ringel and Ringel found gaps in the proofs. The problem is still unsolved. The conjecture states that the number of crossings in an optimal drawing of $K_{m,n}$ is $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. The conjecture is a hard problem that has resisted attempts to solve it, so we restrict this problem to only consider rectilinear drawings of $K_{m,n}$. That is, those drawings where the edges are straight lines. We study a restriction further Zarankiewicz's Conjecture by allowing only rectilinear drawings where the two partitions of the vertex set can be seperated by a straight line.

Another interest in the area of crossing numbers is to show if two different kinds of crossing numbers differ or not. We also established during this research that there is a 4-partite graph whose rectilinear crossing number and crossing numbers differ.

This project is still an active collaboration between me and the other participants of the MRC program.

5. Conclusion

My continuing interests include many problems in graph theory, network science, and algebraic graph theory. I have a variety of open problems to pull from which are well suited to undergraduate research, and I look forward continuing work with my collaboration network as well as beginning new projects with my new colleagues.

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